PMM U.S.S.R.,Vol.45,pp. 714-717 0021-8928/82/5 0714 $7.50/0

UDC 532.5

ON THE EXISTENCE OF SOLUTION OF THE PROBLEM OF POTENTIAL FLOW OVER A SYMMETRIC PROFILE IN A CHANNEL*

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The plane problem of heavy fluid potential flow over a smooth convex profile symmetric about the vertical axis in a channel with horizontal bottom is considered. Existence of such flow at fairly high Froude numbers is proved. When the profile length approaches zero with the Froude number remaining higher than unity, a uniform flow obtains at the limit.

Similar results were obtained in the case of flow in a channel with a curvilinear bottom in the absence of a profile in the channel in /1—3/, for the flow over a vortex in a channel /4/ and, also, for the flow over a profile in a channel in /5/, where, unlike in the present investigation, not the profile itself, but its image in some parametric plane was specified.

The flow in plane $z = x + iy$ is assumed symmetric about the profile axis of symmetry y and the x axis coincides with the channel bottom. Angle $\Psi(s)$ between the x axis and the tangent to the profile is assumed known and defined as a function of the dimensionless quantity $s = l/L$, where l is the arc coordinate on the profile measured from its upper point on the y axis. When moving along the profile in a clockwise direction, s increases. If the unknown profile length is $2L$, then $0 \leqslant s \leqslant 2$. Owing to the symmetry and convexity /of the profile/ $\Psi(0) = 0$, $\Psi(1) = -\pi$ and $d\Psi/ds \leqslant 0$. The dimensionless curvature $d\Psi/ds$ is bounded. Region G the image of the flow region in the plane of the complex potential $w = \varphi + i\psi$ is also assumed known: it is band $0 \leqslant \psi \leqslant Q$ with a slit along segment $|\varphi| \leqslant mQ$, $\psi = Q(1-q)$, where $m > 0$, $0 < q < 1$. Finally, velocity v_0 at infinity and the acceleration of gravity g are also known.

We map G onto region G_1 in the plane of variable $\zeta = \xi + i\eta$, using function $w = Q(\zeta + 1)$, and G_1 is the band $-1 \leqslant \eta \leqslant 0$ with slit along the segment $|\xi| \leqslant m$, $\eta = -q$. In addition we map the right-hand halves of G and G_1 onto the half-plane $v \geqslant 0$ of variable $U = u + iv$, where the image $U(\zeta)$ satisfies the conditions

$$U(+\infty) = \infty,\ U(0) = -a < -1,\ U(-i) = b > 1,\ U(m - iq) = c \in (-1, 1),\ U[(q \pm 0)i] = \mp 1.$$

By the Christoffel— Schwarz formula

$$\frac{d\zeta}{dU} = \frac{1}{\pi} F(U), \quad F(U) = (U - c)[(U + a)(U^2 - 1)(U - b)]^{-1/2} \tag{1}$$

$$\int_{-a}^{-1} f(u)\, du = \pi q, \quad \int_{-1}^{c} f(u)\, du = \int_{c}^{1} f(u)\, du = \pi m \tag{2}$$

$$(f(u) = |F(u)|)$$

We represent dw/dz in the form $dw/dz = v_0 e^{\omega}$, $\omega = \tau + i\theta$. Let $\omega = \omega_1 + \omega_2$, where $\omega_k = \tau_k + i\theta_k$ $(k = 1, 2)$, and at the free boundary $\tau_1 = \tau$, $\tau_2 = 0$, along the profile $\theta_1 = 0$, $\theta_2 = \theta$, and at the channel bottom and the axis of symmetry $\theta_1 = \theta_2 = 0$. We shall consider besides functions $\omega_k(U)$, also functions $\omega_k(\zeta)$. On the image of the right-hand half of the profile as $|u| < 1$ we evidently have $\theta_2(u) = \Phi[s(u)]$, where $\Phi(s) = -\Psi(s)$, when $0 \leqslant s \leqslant s_0$, and $\Phi(s) = -\Psi(s) - \pi$ when $s_0 < s < 1$, with $s_0 = s(c)$ the dimensionless arc coordinate of the stream run-off point, $s(u)$ is the unknown function, and $s(-1) = 0$, $s(1) = 1$. From the relationships $dw/dz = v_0 e^{\omega}$, $dw/d\zeta = Q$ and (1) with $|u| < 1$ we have

$$s(u) = \gamma \int_{-1}^{u} f(u) \exp[-\tau_1(u) - \tau_2(u)]\, du \tag{3}$$

$$\gamma = \left\{ \int_{-1}^{1} f(u) \exp[-\tau_1(u) - \tau_2(u)]\, du \right\}^{-1} \quad \left(\gamma = \frac{Q}{\pi v_0 L}\right) \tag{4}$$

*Prikl.Matem.Mekhan.,45,No.5,948-952,1981

We denote $d\tau_1(\xi)/d\xi$ by $\mu(\xi)$. The Bernoulli equation and the assumption that velocity at infinity is equal v_0 imply that

$$\mu(\xi) = -\lambda \sin[\theta_1(\xi) + \theta_2(\xi)] \left\{1 + 3\lambda \int_\xi^\infty \sin[\theta_1(\xi) + \theta_2(\xi)]\, d\xi\right\}^{-1}, \quad \tau_1(\xi) = \int_\infty^\xi \mu(\xi)\, d\xi, \quad \lambda = \frac{gQ}{v_0^3} \tag{5}$$

and from the Keldysh-Sedov formula we have

$$\tau_2(u) = \int_{-1}^{1} S(t, u)\, \Phi[s(t)]\, dt, \quad |u| \leqslant 1 \tag{6}$$

$$\theta_2(\xi) = \int_{-1}^{1} S[t, r(\xi)]\, \Phi[s(t)]\, dt, \quad \xi \geqslant 0, \quad r(\xi) \leqslant -a \tag{7}$$

$$\tau_1(u) = \int_{-a}^{-\infty} S(t, u)\, \tau_1[p(t)]\, dt, \quad |u| \leqslant 1, \quad p(t) \geqslant 0 \tag{8}$$

$$S(t, u) = \pi^{-1}(t-u)^{-1} \,|\,(a+u)(a+t)t^{-1}\,|^{1/2}$$

where function $u = r(\xi)$ is the inverse of function $\xi = p(u)$ determined from (1) when $u \leqslant -a$ by formula

$$p(u) = \frac{1}{\pi} \int_u^{-a} f(u)\, du \tag{9}$$

Integrating by parts the Keldysh-Sedov formula with allowance for (9) and $\xi \geqslant 0$ we have

$$\theta_1(\xi) = \frac{1}{\pi^2} \int_{-\infty}^{-a} \mu[p(t)]\, f(t)\, D[t, r(\xi)]\, dt \tag{10}$$

$$D(t, u) = \ln\,|\,[\rho(t) + \rho(u)][\rho(t) - \rho(u)]^{-1}\,|, \quad \rho(t) = |\,a + t\,|^{1/2}$$

Equalities (3) and (5) with relations (4) and (6) — (10) constitute a nonlinear system of integral equations with respect to functions $\mu(\xi)$ $(\xi \geqslant 0)$ and $s(u)$ $(|u| \leqslant 1)$. Below, using the Schauder principle we prove the existence of solution of that system under specific constraints on the Froude number $1/\lambda$ that depend on the quantities m, q.

Let $x = \{\mu(\xi), s(u)\}$, H_α be the Hölder function in the space $[-1, 1]$ with index α and norm $\|s\|_\alpha$, C be the space of functions continuous in $[0, \infty]$, and $E = C \times H_\alpha$, $E_0 = E_0(N, \beta, R)$ be the closed subset of elements from E_0 that satisfy the conditions

$$|\mu(\xi)| \leqslant N e^{-\pi\beta\xi/2}, \quad \|s\|_\alpha \leqslant R$$
$$s(-1) = 0, \quad s(1) = 1, \quad [s(u') - s(u')](u'' - u') \geqslant 0 \tag{11}$$

The investigated system of equations may be written in the operational form: $x = Ax$. Let us hsow that for some values of N, β, R operator A converts E_0 into itself. Let $x_1 = \{\mu_1(\xi), s(u)\} \in E_0$. We obtain estimates of $x_2 = Ax_1$ in several stages.

1°. The right-hand side of (3) $s_2(u)$ increases, and by virtue of (4) $s_2(-1) = 0$, $s_2(1) = 1$.

2°. Since for $u < -a$ we have $f(u) \leqslant [(1+u)(a+u)]^{-1/2}$, hence from (9) we obtain

$$\xi = p(u) < \pi^{-1} \ln\,|\,(4u + 3a + 1)(a-1)^{-1}\,|, \quad u = r(\xi) < \frac{1}{4}(1-a)\, e^{\pi\xi} \tag{12}$$

3°. Since $-\pi \leqslant \Psi(s) \leqslant 0$, hence $|\Phi(s)| \leqslant \pi$ and from (7) we obtain the inequality $|\theta_2(\xi)| < 4\,|\,ar(\xi)\,|^{-1/2}$, and by virtue of (12)

$$|\theta_2(\xi)| < k_1 e^{-\pi\xi/2}, \quad k_1 = 8\,[a(a-1)]^{-1/2} \tag{13}$$

4°. We continue $\mu_1(\xi)$ on the semiaxis $\xi < 0$ in the odd manner. The right-hand side of (10) $\theta_1(\xi)$ is, then, the limit value of function $\theta_1(\xi)$ which for $\zeta = \xi$ is harmonic in the doubly-connected region G_1 and vanishes when $\zeta = \xi - i$ and at the slit edges and, for $\zeta = \xi$, has a normal derivative equal to $\mu_1(\xi)$. Using (11) and the maximum principle we can show that $|\theta_1(\xi)| \leqslant \varkappa(\xi)$, where $\varkappa(\xi)$, is the limit value when $\zeta = \xi$ of a function that is harmonic in the simply connected region $(-1 \leqslant \eta \leqslant 0,\ |\xi| < \infty)$ and vanishes when $\zeta = \xi - i$ and has a normal derivative $\delta(\xi) = Ne^{-\pi\beta|\xi|/2}$ when $\zeta = \xi$. Assuming that $0 < \beta < 1$ and using Lemma 1 from /3/, we obtain the inequality $|\theta_1(\xi)| < (1-\beta^2)^{-1}\delta(\xi)$. This and (13) imply that the right-hand side of (5) $\mu_2(\xi)$ satisfies the inequality $|\mu_2(\xi)| < \delta(\xi)$, when $\xi > 0$, if

$$1 - 6\lambda B(\pi\beta)^{-1} > 0, \quad \lambda B[1 - 6\lambda B(\pi\beta)^{-1}]^{-1} \leqslant N, \quad (B = N/(1-\beta^2) + k_1) \tag{14}$$

Analysis shows that when

$$\lambda \leqslant [(1-\beta^2)^{-1/2} + (6k_1/(\pi\beta))^{1/2}]^{-2} \tag{15}$$

then the following statements hold:

$$N_0 = \frac{\pi\beta(1-\beta^2)}{12}\left\{\frac{1}{\lambda} - \frac{1}{1-\lambda^2} - \frac{6k_1}{\beta\pi} - \left[\left(\frac{1}{\lambda} - k_2\right)\left(\frac{1}{\lambda} - k_3\right)\right]^{1/2}\right\} \tag{16}$$

$$(k_{2,3} = [(1-\beta^2)^{-1/2} \pm (6k_1/(\pi\beta))^{1/2}]^2)$$

really and positively; inequalities (14) are satisfied for $N = N_0$, while the second of them is not satisfied when $N < N_0$. Since $0 < \beta < 1$, (15) is the corollary of the inequality

$$\lambda \leqslant [(1-\beta)^{-1} + (6k_1/\pi)^{1/2}\beta^{-1}]^{-2} = \nu(\beta) \tag{17}$$

The maximum value of $\nu(\beta)$ is attained for $\beta = \beta_0$, where

$$\beta_0 = (1 + (6k_1/\pi)^{-1/4}]^{-1}, \quad \nu(\beta_0) = [1 + (6k_1/\pi)^{1/4}]^{-4} \tag{18}$$

Subsequently we shall consider that $\beta = \beta_0$, $N = N_0$ and the inequality (17) is fulfilled.

5°. By virtue of inequality (11) and formula (8), $|\tau_1(u)| \leqslant \max|\tau_1(\xi)| < 2N_0/(\pi\beta) = k_4$ when $|u| \leqslant 1$. Let $f_1(u) = |u - c|(1-u^2)^{-1/2}$. Then $|u| \leqslant 1$ is

$$f_1(u)[(a+1)(b+1)]^{-1/2} \leqslant f(u) \leqslant f_1(u)[(a-1)(b-1)]^{-1/2} \tag{19}$$

Using (4), (19), $|\tau_1(u)| < k_4$, the Jensen inequality /6/, and the inequality

$$\int_{-1}^{1} f(u)\,du > 2$$

we obtain

$$\gamma < \frac{1}{2}[(a+1)(b+1)]^{1/2}\exp\left[k_4 - \frac{1}{2}\int_{-1}^{1}\tau_2(u)\,du\right]$$

where the last integral is estimated using formulas (6) with allowance for the inequality $|\Phi(s)| \leqslant \pi$. As the result we have

$$\gamma < 4^{-1}\exp[k_4 + 2(1+\sqrt{2})][(a+1)(b+1)]^{1/2} \tag{20}$$

6°. Integrating (6) by parts and taking into account that $\Phi(s_0 - 0) - \Phi(s_0 + 0) = \pi$ when $|u| \leqslant 1$ we have

$$\tau_2(u) = \ln\frac{|u-c|}{[(c+a)^{1/2} + (u+a)^{1/2}]^2} - \frac{1}{2\pi}\int_{-1}^{1}\frac{d\Psi[s(t)]}{dt}\cdot D(t,u)\,dt \tag{21}$$

Taking into account that

$$D(t,u) > \ln[2(a-1)], \quad \Psi(0) - \Psi(1) = \pi, \quad |\tau_1(u)| < k_4$$

and (19) and (20), from (3) and (21) we have

$$\frac{ds_2}{du} < k_5(1-u^2)^{-1/2}, \qquad k_5 = \frac{1}{2}\exp[2(k_4 + 1 + \sqrt{2})] \times \left(\frac{a+1}{a-1}\right)^{1/2}\left(\frac{b+1}{b-1}\right)^{1/2}$$

Hence for any $u', u'' \in [-1, 1]$

$$|s_2(u'') - s(u')| \leqslant 2k_5|u'' - u'|^{1/2}, \quad \|s_2\|_{1/2} < 1 + 2k_5 \tag{22}$$

The above estimates show that the operator A transforms into itself the closed set $E_0(N, \beta, R)$ from the space $E = C \times H_\alpha$ ($\alpha < 1/2$, $\beta = \beta_0$ is determined in (18), and $N = N_0$ is from (16), $R = 1 + k_5\sqrt{2}$) when λ satisfied inequality (17). The complete continuity of operator A on E_0 by the norm of space E is proved by conventional methods. By Schauder's theorem equation $x = Ax$ has at least one solution in E_0 when the equivalent to (17) and (18) inequality

$$\lambda \leqslant \{1 + 2\,[a\,(a-1)\pi^2/9]^{-1/2}\}^{-1} = \lambda_0 \tag{23}$$

is satisfied.

Parameters a, b, c are determined by the system of Eqs.(2) and depend on the input parameters $q \in (0, 1)$ and $m > 0$. For an effective check of the fulfillment of condition (23) we provide the estimate of the lower bound of the quantity $(a-1)$.

We change the variable in (2) using formula $u = (a+b)t - a$. To quantities $u = a, -1, c, 1, b$ correspond $t = 0, t_1, t_2, t_3, 1$ $(0 < t_k < 1)$ that satisfy the system of equations

$$\pi q = \int_0^{t_1} f_2(t)\,dt, \quad \pi m = \int_{t_1}^{t_2} f_2(t)\,dt = \int_{t_2}^{t_3} f_2(t)\,dt \tag{24}$$
$$(f_2(t) = |\,t_2 - t\,|\;|\,t\,(t-t_1)(t-t_3)(t-1)\,|^{-1/2})$$

and $a - 1 = 2t_1\,(t_3 - t_1)^{-1}$.

For $t \in (t_1, t_2)$ we have $f_2(t) > (t_2 - t)[(t-t_1)(t_3-t)]^{-1/2}$; assuming that $t_3 - t_2 \leqslant t_2 - t_1$ we obtain from (24) $t_2 - t_1 < 2\pi m$. Hence $t_3 - t_1 < 4\pi m$.

For $t \in (0, t_1)$ we have $f_2(t) < t_2^{1/2}\,[t\,(1-t_2)(t_1-t)]^{-1/2}$, and from (24) $q < [t_2\,(1-t_2)^{-1}]^{1/2}$, hence $t_2 > q^2/2$.

For $t \in (t_1, t_2)$ we have $f_2(t) > t_2/t - 1$, and by virtue of (24) $\pi m > t_2[\ln(t_2/t_1) - 1]$; this with the estimate $t_2 > q^2/2$ shows that $t_1 > (q^2/2)\exp[-(1 + 2\pi m/q^2)]$.

From this inequality and the inequality $t_3 - t_1 < 4\pi m$ we obtain the required estimate

$$a - 1 > q^2\,(4\pi m)^{-1}\exp[-(1 + 2\pi m/q^2)] \tag{25}$$

Taking into account (23) and (25) we conclude that $\lambda_0 \to 0$ as $m/q^2 \to \infty$, and $\lambda_0 \to 1$ as $m/q^2 \to 0$. Using the lower bound of γ it is possible to show that the last condition is equivalent to $Lv_0/Q \to 0$. The obtained estimates make it, moreover, possible to prove that $N_0 \to 0$ as $m/q^2 \to 0$, i.e. that a uniform flow is obtained.

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Translated by J.J.D.